Quantifying Fragility of Network-Coupled Oscillators and Electric Power Grids with Resistance Distances.

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MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
MT,Pagnier and Jacquod submitted (2018), arXiv:1810.09694.
MT and Jacquod in preparation (2018).

Motivation

Complex network-coupled dynamical systems:

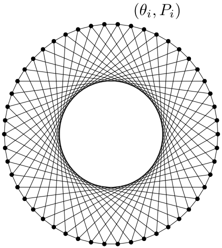
- **Individual Units**:
  - Degrees of freedom $\to (\theta_{i,1}, \theta_{i,2}, \theta_{i,3}, ...)$.
  - Internal parameters $\to (P_{i,1}, P_{i,2}, P_{i,3}, ...)$.
- **Complex Network**:
  - Coupling b_{ij} between units i and j.

Perturbation

- $(P_{i,1}, P_{i,2}, ...) \to (P_{i,1} + \delta P_{i,1}, P_{i,2} + \delta P_{i,2}, ...)$.

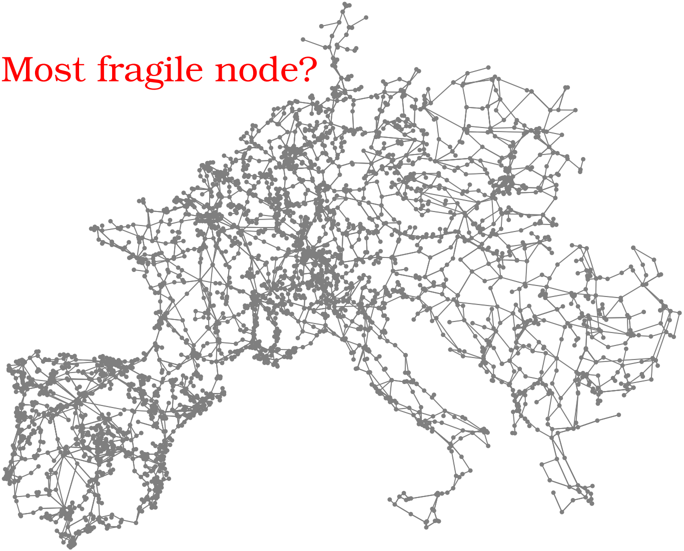
$\to$ How does the response of $(\theta_{i,1}, \theta_{i,2}, ...)$ depend on the coupling network?

Most fragile network?



MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120** 084101 (2018).

Interesting Question about Coupled Dynamical Systems 2



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Swing Equations in the lossless line limit (second-order Kuramoto):

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j b_{ij}\sin(\theta_i - \theta_j) \;\;,\;\; i = 1, ..., n. \tag{1}$$

$b_{ij} = b_{ji} \geq 0$.

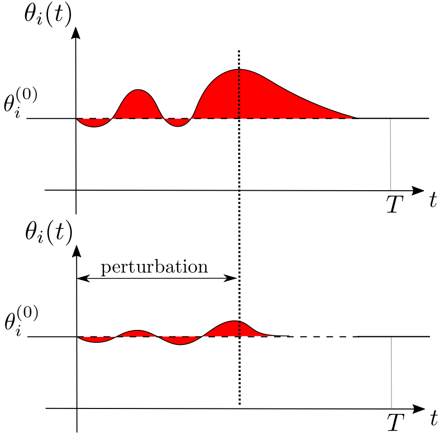
Steady-state solutions: Synchronous state $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j b_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \;\;,\;\; i = 1, ..., n. \tag{2}$$

$\sum_i P_i = 0.$

Perturbations: $P_i \to P_i^{(0)} + \delta P_i(t)$.

Performance measures :



$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t \,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t \,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty) \,.$$

Noisy disturbances $\to$ divide by T.

Perturbations : $P_i \to P_i^{(0)} + \delta P_i(t)$.

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Response to Perturbations: Linearization

Linear response: Perturbation of the natural frequencies (inj/cons powers).

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \rightarrow \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$m\delta\ddot{\boldsymbol{\theta}}(t) + d\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ , \tag{3}$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & i \neq j \, , \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & i = j \, . \end{cases} \tag{4}$$

Topology $\rightarrow b_{ij}$.
Steady state $\rightarrow \{\theta_i^{(0)}\}$.
Expanding on the eigenvectors $\mathbf{u}_\alpha$ of $\mathbb{L}$, we have $\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha$.
$\rightarrow \mathcal{P}_1(T)$, $\mathcal{P}_2(T)$ for specific perturbations

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)
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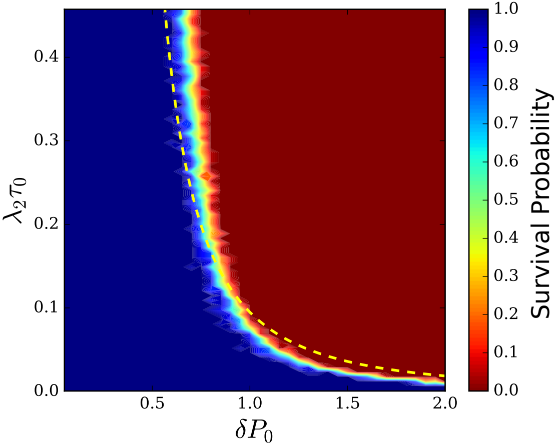
Change of fixed point !



MT,Delabays and Jacquod submitted (2018), arXiv:1812.09497.

Response to Perturbations: Time Scales

Intrinsic Time Scales

- Individual elements: m/d.
- Network relaxation: d/λ_α with $\{\lambda_\alpha\}$ the eigenvalues of $\mathbb{L}$.

Perturbation Time Scale

- Correlation time of the external perturbation $\delta \boldsymbol{P}(t)$.

Noisy perturbations

- $\langle \delta P_i(t) P_j(t') \rangle = \delta P_{0i}^2 \delta_{ij} \exp[-|t-t'|/\tau_0]$.

Correlation time $\rightarrow \tau_0$.

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Performance Measures Asymptotics

Performance Measures for Noisy Perturbations

$$\mathcal{P}_1^\infty = \sum_{\alpha\geq 2} \frac{\sum_i \delta P_{0i}^2 u_{\alpha,i}^2 (\tau_0 + m/d)}{\lambda_\alpha(\lambda_\alpha \tau_0 + d + m/\tau_0)} \, . \tag{5}$$

Short time correlated: $\tau_0 \ll d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty \simeq \frac{\tau_0}{d} \sum_{\alpha\geq 2} \frac{\sum_i \delta P_{0i}^2 u_{\alpha,i}^2}{\lambda_\alpha} \, . \tag{6}$$

Long time correlated: $\tau_0 \gg d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty \simeq \sum_{\alpha\geq 2} \frac{\sum_i \delta P_{0i}^2 u_{\alpha,i}^2}{\lambda_\alpha^2} \, . \tag{7}$$

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Global Robustness & Local Vulnerabilities

Global Robustness:

Averaging over an ergodic ensemble of perturbation vectors, $\tau_0 \ll d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle \tau_0}{d} \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,,$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \langle \delta P_0^2 \rangle \sum_{\alpha \geq 2} \lambda_\alpha^{-2} \,.$$

Local Vulnerability:

Perturbing a specific node k i.e. $\delta P_{0i} = \delta_{ik} \delta P_0$, $\tau_0 \ll d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \frac{\delta P_0^2 \tau_0}{d} \sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha} \,,$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \delta P_0^2 \sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^2} \,.$$

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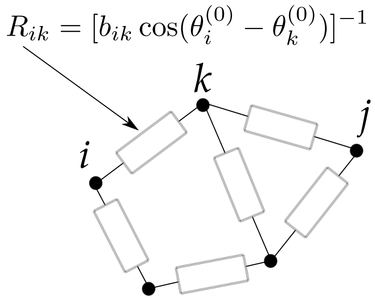
Resistance Distance

$$\Omega_{ij} = \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{jj} - \mathbb{L}^{\dagger}_{ij} - \mathbb{L}^{\dagger}_{ji} = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} \,. \tag{8}$$

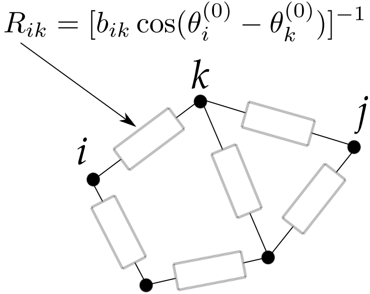
$\mathbb{L}^{\dagger}$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).



Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Kirchhoff Index

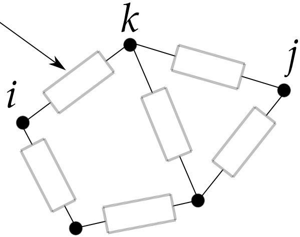
$$Kf_1 = \sum_{i<j} \Omega_{ij} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \, . \tag{9}$$



Resistance Centrality

$$C_1(k) = \left[n^{-1} \sum_j \Omega_{kj} \right]^{-1} = \left[\sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha} + n^{-2} Kf_1 \right]^{-1} . \qquad (10)$$

$$R_{ik} = [b_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Generalized Resistance Distances

$$\Omega_{ij}^{(m)} = \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji} \tag{11}$$

$$= \sum_{\alpha\geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^m} \,, \tag{12}$$

$$\mathbb{L}' = \mathbb{L}^m \,. \tag{13}$$

Generalized Kirchhoff Indices

$$Kf_m = \sum_{i<j} \Omega_{ij}^{(m)} = n \sum_{\alpha\geq 2} \lambda_\alpha^{-m} \,. \tag{14}$$

Generalized Resistance Centralities

$$C_m(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(m)} \right]^{-1} = \left[\sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^m} + n^{-2} Kf_m \right]^{-1} . \tag{15}$$

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)
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Global Robustness:

Averaging over an ergodic ensemble of perturbation vectors, $\tau_0 \ll d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle \tau_0}{nd} Kf_1 \,,$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle}{n} Kf_2 \,.$$

Local Vulnerability:

Perturbing a specific node k i.e. $\delta P_{0i} = \delta_{ik} \delta P_0$, $\tau_0 \ll d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} Kf_1 \right) \,,$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \delta P_0^2 \left(C_2^{-1}(k) - n^{-2} Kf_2 \right) \,.$$

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Summary

Global Robustness:

$\tau_0 \ll d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle \tau_0}{nd} K f_1 \,,$$
$$\langle \mathcal{P}_2^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle \tau_0}{dm} \frac{(n-1)}{n} \,.$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\langle \mathcal{P}_1^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle}{n} K f_2 \,,$$
$$\langle \mathcal{P}_2^\infty \rangle \simeq \frac{\langle \delta P_0^2 \rangle}{nd\tau_0} K f_1 \,.$$

Local Vulnerability:

$\tau_0 \ll d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} K f_1 \right) \,,$$
$$\mathcal{P}_2^\infty(k) \simeq \frac{\delta P_0^2 \tau_0}{dm} \frac{(n-1)}{n} \,.$$

$\tau_0 \gg d/\lambda_\alpha, m/d$

$$\mathcal{P}_1^\infty(k) \simeq \delta P_0^2 \left(C_2^{-1}(k) - n^{-2} K f_2 \right) \,,$$
$$\mathcal{P}_2^\infty(k) \simeq \frac{\delta P_0^2}{d\tau_0} \left(C_1^{-1}(k) - n^{-2} K f_1 \right) \,.$$

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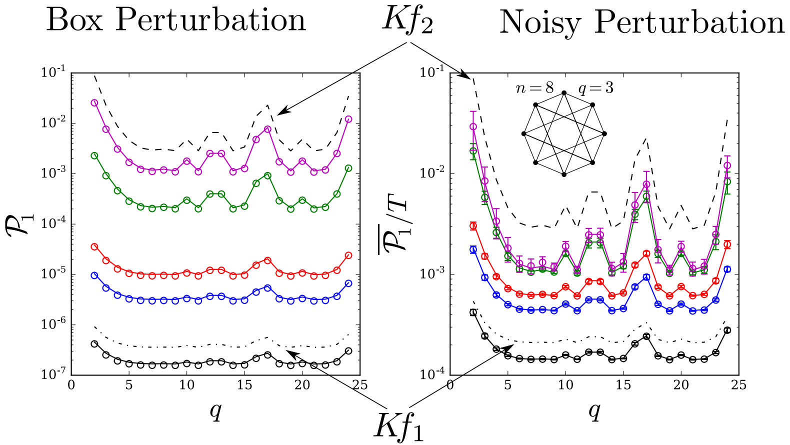
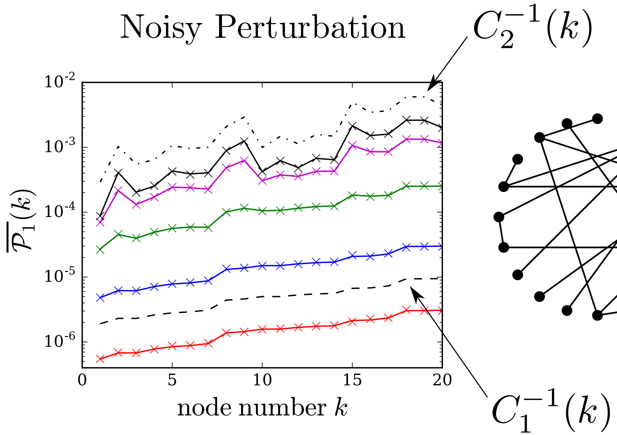
Averaged Global Robustness and Kf_m's



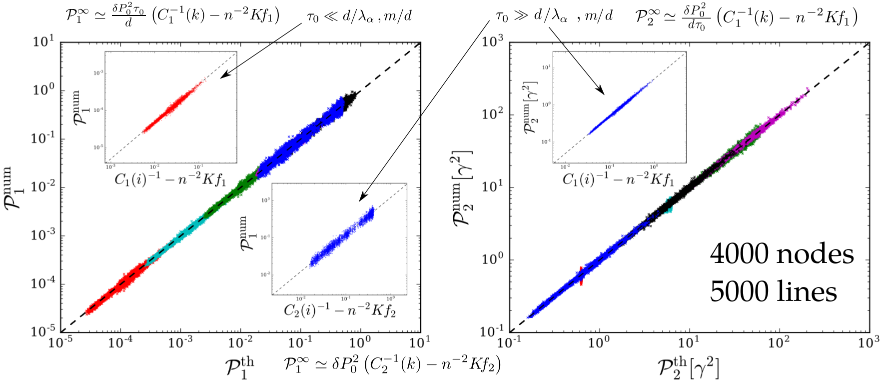
MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).

Specific Local Vulnerabilities and C_m's



MT and Jacquod in preparation (2018).

Specific Local Vulnerabilities and C_m's



MT,Pagnier and Jacquod submitted (2018), arXiv:1810.09694.

Physical Realization : European Electrical Grid

$\tau_0 \ll d/\lambda_\alpha\,, m/d$

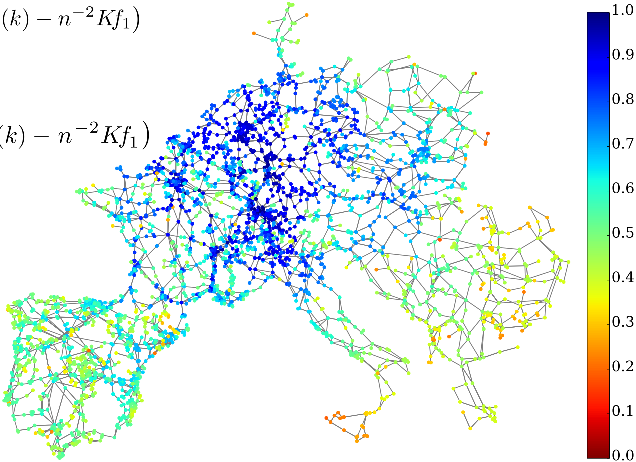
$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0}{d}\left(C_1^{-1}(k) - n^{-2} K f_1\right)$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2 \tau_0}{dm}\frac{(n-1)}{n}$

$\tau_0 \gg d/\lambda_\alpha \quad, m/d$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2}{d\tau_0}\left(C_1^{-1}(k) - n^{-2} K f_1\right)$

$C_1(i)/\max[C_1(i)]$



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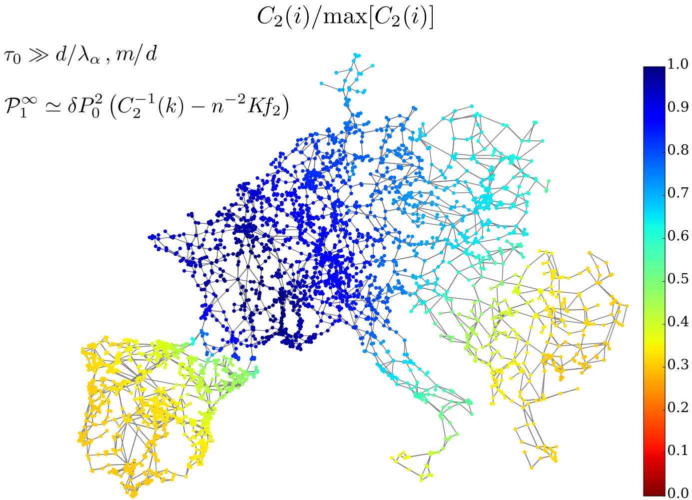
Physical Realization : European Electrical Grid



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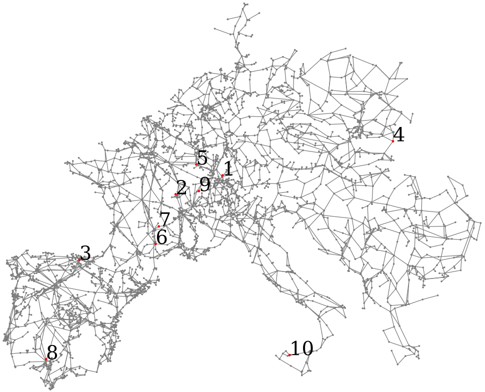
Physical Realization : European Electrical Power Grid



node #	$C_{\rm geo}$	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{\rm num}$	$\mathcal{P}_2^{\rm num}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

MT,Pagnier and Jacquod submitted (2018), arXiv:1810.09694.

Conclusion

Global Robustness

- Generalized Kirchhoff Indices, Kf_m's.

Local Vulnerabilities

- Generalized Resistance Centralities, C_m's.
- Establish a ranking of the nodes.

$\rightarrow$ m depends on which performance measures you are interested in and on the correlation time of the perturbation.

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
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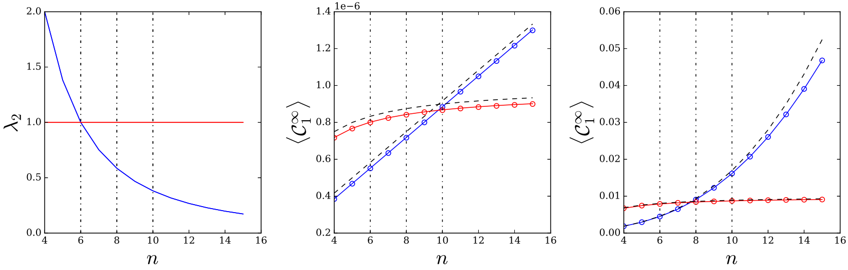
HES-SO Valais-Wallis

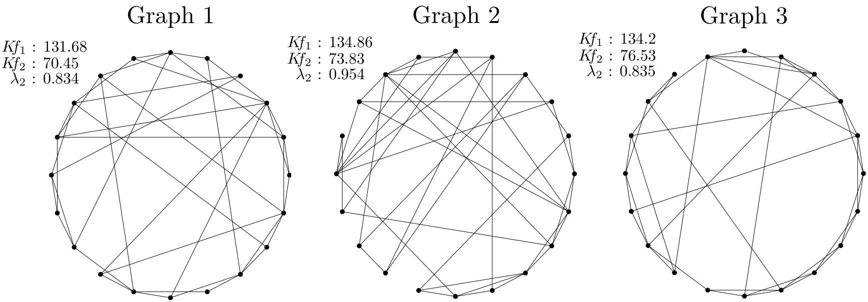
Supplemental Material



blue : cycle graph
red : star graph

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Supplemental Material



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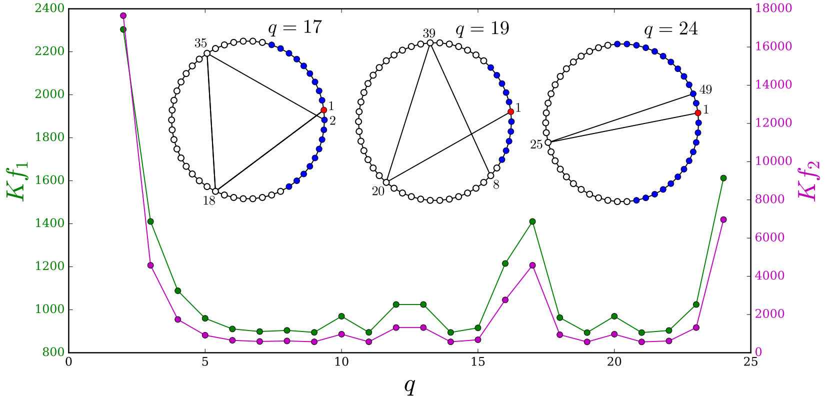

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