Quantifying Vulnerabilities of Complex Oscillatory Networks

Melvyn Tyloo

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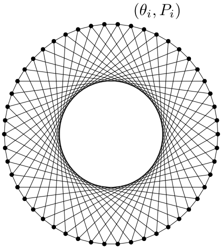
August 26, 2019

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances* (2019), arXiv:1810.09694.
MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

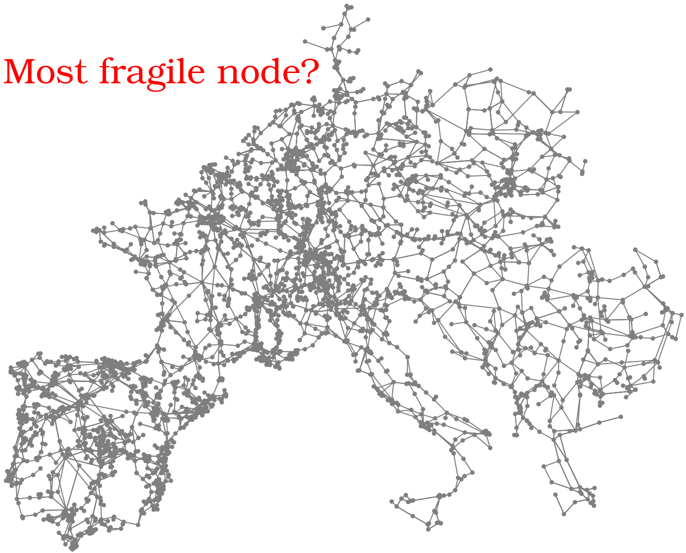
Interesting Question about Coupled Dynamical Systems 1

Most fragile network?



MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120** 084101 (2018).

Interesting Question about Coupled Dynamical Systems 2



MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Complex Networks Metrics

Centrality Measures (Local)

- Geodesic Distance
- PageRank
- Katz centrality
- Harmonic
- Degree
- Communicability
- Betweenness Centrality
- ...

Graph Measures (Global)

- Average Geodesic Distance
- Degree Heterogeneity
- Average Degree
- Degree Distribution: Scale-Free...
- Clustering Coefficient
- A,Di-ssortativity
- ...

P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).

Complex Networks Metrics $\leftrightarrow$ Coupled Dynamical System

Centrality Measures (Local)

- Geodesic Distance
- PageRank $\rightarrow$ RWs on complex networks.
- Katz centrality
- Harmonic/Closeness
- Degree
- Communicability $\rightarrow$ R diff. processes on complex networks.
- Betweenness Centrality
- ...

Graph Measures (Global)

- Average Geodesic Distance
- Degree Heterogeneity
- Average Degree
- Degree Distribution: Scale-Free... $\tilde{\rightarrow}$ SIS transition.
- Clustering Coefficient
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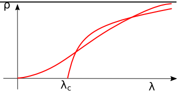
P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).

Complex Networks Metrics ↔ Coupled Dynamical System

Degree Distribution: Scale-Free... $\tilde{\rightarrow}$ SIS transition.

VOLUME 86, NUMBER 14 PHYSICAL REVIEW LETTERS 2 APRIL 2001



Epidemic Spreading in Scale-Free Networks

Romualdo Pastor-Satorras[1] and Alessandro Vespignani[2]
[1]*Departament de Física i Enginyeria Nuclear, Universitat Politècnica de Catalunya, Campus Nord, Mòdul B4, 08034 Barcelona, Spain*
[2]*The Abdus Salam International Centre for Theoretical Physics (ICTP), P.O. Box 586, 34100 Trieste, Italy*
(Received 20 October 2000)

Communicability → R diff. processes on complex networks.

PHYSICAL REVIEW E **77**, 036111 (2008)

Communicability in complex networks

$$G_{pq} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}^k)_{pq}}{k!} = (e^{\mathbf{A}})_{pq}.$$

Ernesto Estrada[1,*] and Naomichi Hatano[2]
[1]*Complex Systems Research Group, X-rays Unit, RIAIDT, Edificio CACTUS, University of Santiago de Compostela, 15076 Santiago de Compostela, Spain*
[2]*Institute of Industrial Science, University of Tokyo, 4-6-1 Komaba, Meguro, Tokyo 153-8505, Japan*
(Received 21 August 2007; published 11 March 2008)

PageRank $\rightarrow$ RWs on complex networks.

The PageRank Citation Ranking: Bringing Order to the Web

L. Page, S. Brin, R. Motwani and T. Winograd

January 29, 1998

Complex Networks Metrics ↔ Coupled Dynamical System

Centrality Measures (Local)

- Geodesic Distance
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- Katz centrality
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- ...

Graph Measures (Global)

- Average Geodesic Distance
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- Clustering Coefficient
- A,Di-ssortativity
- ...

Not related to particular network dynamics... can be misleading.

P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).

P. Hines, E. Cortilla-Sanchez and S. Blumsack, Chaos **22**, 033122 (2010).

Complex Network-Coupled Dynamical Systems

Nodes

- **Individual Units**:
  - Degrees of freedom $\rightarrow (\theta_{i,1}, \theta_{i,2}, \theta_{i,3}, ...)$.
  - Internal parameters $\rightarrow (P_{i,1}, P_{i,2}, P_{i,3}, ...)$.

Edges

- **Complex Network**:
  - Coupling b_{ij} between units i and j $\rightarrow$ adjacency matrix of the complex network.
  - Coupling function of the degrees of freedom.

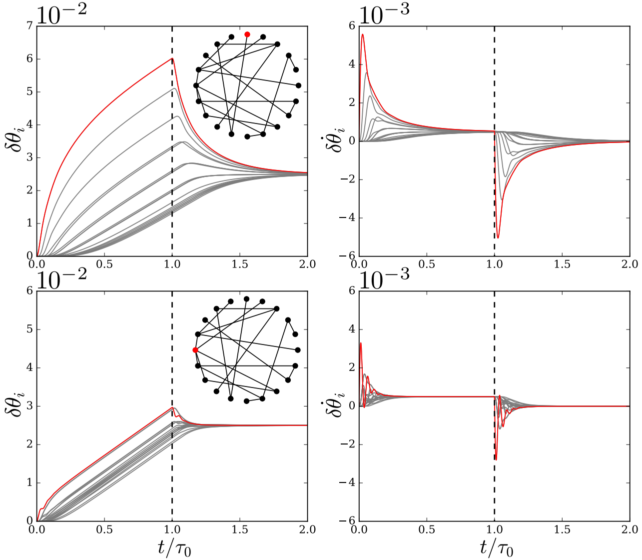
Perturbation

- $(P_{i,1}, P_{i,2}, ...) \rightarrow (P_{i,1} + \delta P_{i,1}, P_{i,2} + \delta P_{i,2}, ...)$.

$\rightarrow$ Quantify the transient dynamics.

$\rightarrow$ How does the response of $(\theta_{i,1}, \theta_{i,2}, ...)$ depend on the coupling network?

Coupled Dynamical Systems: Example



MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

Swing Equations in the lossless line limit (second-order Kuramoto):

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j b_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$b_{ij} = b_{ji} \geq 0$.

Steady-state solutions: Synchronous state $\{\theta_i^{(0)}\}$ such that:

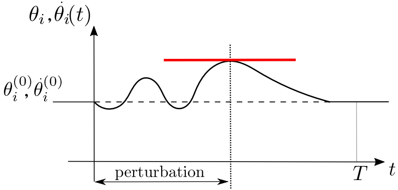
$$P_i = \sum_j b_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n.$$

$\sum_i P_i = 0.$

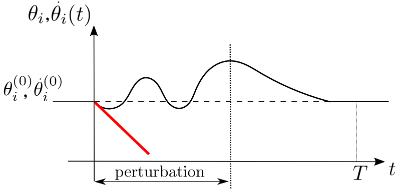
Perturbations: $P_i \to P_i^{(0)} + \delta P_i(t)$.

Quantifying Robustness

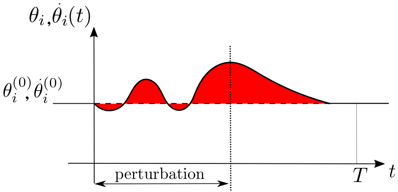
- Maximum of the response, $\max_t(\theta_i)$.



- Rate of change of frequency (RoCoF), $\ddot{\theta}_i$.

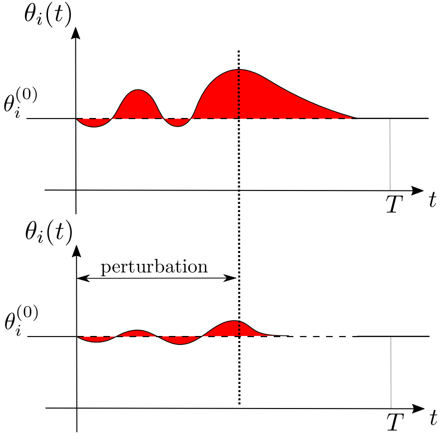


- Performance measure (quadratic integrals over the transient).



Quantifying Robustness

Performance measures :



$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t \,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t \,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty) \,.$$

Noisy disturbances $\to$ divide by T.

Perturbations : $P_i \to P_i^{(0)} + \delta P_i(t)$.

Response to Perturbations: Linearization

Linear response: Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \to \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$m\delta\ddot{\boldsymbol{\theta}}(t) + d\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & i \neq j \, , \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & i = j \, . \end{cases}$$

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Topology $\rightarrow b_{ij}$.
Steady state $\rightarrow \{\theta_i^{(0)}\}$.

Response to Perturbations: Linearization

Linear response

$$m\delta\ddot{\boldsymbol{\theta}}(t) + d\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

Expanding on the eigenvectors $\mathbf{u}_\alpha$ of $\mathbb{L}$, we have $\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha$.

$$c_\alpha(t) = m^{-1}\, e^{-(\gamma+\Gamma_\alpha)t/2} \int_0^t e^{\Gamma_\alpha t_1} \int_0^{t_1} \delta\boldsymbol{P}(t_2)\cdot\mathbf{u}_\alpha\, e^{(\gamma-\Gamma_\alpha)t_2/2}\,\mathrm{d}t_2\mathrm{d}t_1 \ ,$$

$\gamma = d/m$ and $\Gamma_\alpha = \sqrt{\gamma^2 - 4\lambda_\alpha/m}$. $\rightarrow \mathcal{P}_1$, $\mathcal{P}_2$ for specific perturbations,

$$\mathcal{P}_1^\infty = \sum_{\alpha\geq 2} \int_0^\infty c_\alpha^2(t)\mathrm{d}t \ ,$$
$$\mathcal{P}_2^\infty = \sum_{\alpha\geq 2} \int_0^\infty \dot{c}_\alpha^2(t)\mathrm{d}t \ .$$

Performance Measures: Alternative Approaches

Transfer functions

Global performance metrics for synchronization of heterogeneously rated power systems: The role of machine models and inertia

Fernando Paganini, *Fellow, IEEE*, and Enrique Mallada, *Member, IEEE*

$$g_i(s) = \frac{1}{m_i s^2 + d_i s}.$$

Observability Gramian

456 IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS, VOL. 5, NO. 1, MARCH 2018

Performance Measures for Linear Oscillator Networks Over Arbitrary Graphs

Theodore W. Grunberg, *Student Member, IEEE*, and Dennice F. Gayme, *Senior Member, IEEE*

$$X := \int_0^\infty e^{A^\intercal \tau} C^\intercal C e^{A\tau} d\tau$$

Response to Perturbations: Time Scales

Intrinsic Time Scales

- Individual elements: m/d.
- Network relaxation: d/λ_α with $\{\lambda_\alpha\}$ the eigenvalues of $\mathbb{L}$.

Perturbation Time Scale

- Correlation time of the external perturbation $\delta \boldsymbol{P}(t)$.

Quench perturbations

- $\delta P_i(t) = \delta P_{0i} \Theta(t) \Theta(\tau_0 - t)$.

Quench duration $\rightarrow \tau_0$.

Noisy time correlated perturbations

- $\langle \delta P_i(t) P_j(t') \rangle = \delta P_{0i}^2 \delta_{ij} \exp[-|t - t'|/\tau_0]$.

Correlation time $\rightarrow \tau_0$.

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.
MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

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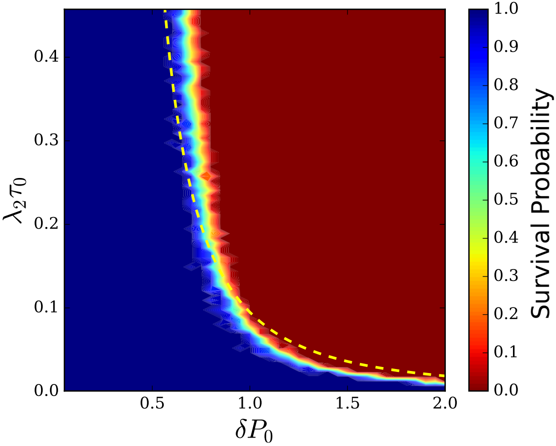
(Response to Large Perturbations)

Change of fixed point !



MT,Delabays and Jacquod *Phys. Rev. E* **99**, 062213 (2019).

Performance measures for quench perturbations

$$
\begin{aligned}
\mathcal{P}_1^\infty &= \frac{m}{8\gamma}\sum_{\alpha\geq 2}\frac{(\delta \boldsymbol{P}_0\cdot \mathbf{u}_\alpha)^2}{\Gamma_\alpha\lambda_\alpha^3}\Big[2\Gamma_\alpha(4\gamma\tau_0\lambda_\alpha/m-3\gamma^2-\Gamma_\alpha^2)\\
&+ (\gamma+\Gamma_\alpha)^3 e^{-\tau_0\frac{(\gamma-\Gamma_\alpha)}{2}} - (\gamma-\Gamma_\alpha)^3 e^{-\tau_0\frac{(\gamma+\Gamma_\alpha)}{2}}\Big]\,,\\
\mathcal{P}_2^\infty &= \frac{1}{2d}\sum_{\alpha\geq 2}\frac{(\delta \boldsymbol{P}_0\cdot \mathbf{u}_\alpha)^2}{\Gamma_\alpha\lambda_\alpha}\Big[2\Gamma_\alpha-(\gamma+\Gamma_\alpha)e^{-\frac{\tau_0(\gamma-\Gamma_\alpha)}{2}}\\
&+ (\gamma-\Gamma_\alpha)e^{-\frac{\tau_0(\gamma+\Gamma_\alpha)}{2}}\Big]\,.
\end{aligned}
$$

MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

Performance Measures Asymptotics

Short duration: $\tau_0 \ll m/d,\ (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty = \frac{\tau_0^2}{2d} \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha} \,,$$

$$\mathcal{P}_2^\infty = \frac{\tau_0^2}{2md} \sum_{\alpha \geq 2} (\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2 \,.$$

Long duration: $\tau_0 \gg m/d,\ (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty = \tau_0 \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha^2} \,,$$

$$\mathcal{P}_2^\infty = d^{-1} \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha}$$

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
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Local Vulnerability:
Perturbing a specific node k i.e. $\delta P_{0i} = \delta_{ik}\delta P_0$,
$\tau_0 \ll m/d$, $(\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2d} \sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha} ,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2md} \sum_{\alpha\geq 2} u_{\alpha,k}^2 ,$$

$\tau_0 \gg m/d$, $(\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \delta P_0^2 \tau_0 \textstyle\sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^2} ,$$
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Global Robustness:
Averaging over an ergodic ensemble of perturbation vectors,
$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$
\begin{aligned}
\langle \mathcal{P}_1^\infty \rangle &= \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2d} \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,, \\
\langle \mathcal{P}_2^\infty \rangle &= \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2md} \frac{n-1}{n} \,.
\end{aligned}
$$

$\tau_0 \gg m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$
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\langle \mathcal{P}_2^\infty \rangle &= \frac{\langle \delta P_0^2 \rangle}{d} \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,.
\end{aligned}
$$

Global Robustness & Local Vulnerabilities

Local Vulnerability:
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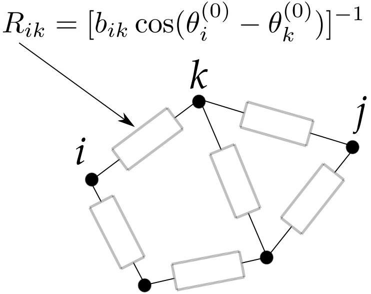
$$\langle\mathcal{P}_1^\infty\rangle = \langle\delta P_0^2\rangle \tau_0 \sum_{\alpha\geq 2} \lambda_\alpha^{-2}\,,$$

$$\langle\mathcal{P}_2^\infty\rangle = \frac{\langle\delta P_0^2\rangle}{d} \sum_{\alpha\geq 2} \lambda_\alpha^{-1}\,.$$

Resistance Distance

$$\Omega_{ij} \;=\; \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{jj} - \mathbb{L}^{\dagger}_{ij} - \mathbb{L}^{\dagger}_{ji} = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} \,.$$

$\mathbb{L}^{\dagger}$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).

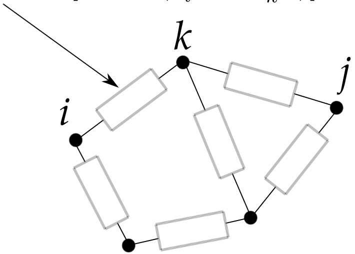


Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,.$$

$$R_{ik} = [b_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$

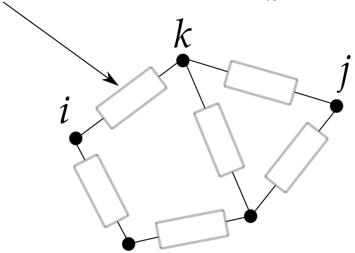


Resistance Distances, $Kf_p's$ and C_p's

Resistance Centrality

$$C_1(k) = \left[n^{-1}\sum_j \Omega_{kj}\right]^{-1} = \left[\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha} + n^{-2}Kf_1\right]^{-1}.$$

$$R_{ik} = [b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Generalized Resistance Distances

$$\begin{aligned}\Omega_{ij}^{(p)} &= \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji} \\ &= \sum_{\alpha\geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^p}\,, \\ \mathbb{L}' &= \mathbb{L}^p\,.\end{aligned}$$

Generalized Kirchhoff Indices

$$Kf_p = \sum_{i<j} \Omega_{ij}^{(p)} = n \sum_{\alpha\geq 2} \lambda_\alpha^{-p}\,.$$

Generalized Resistance Centralities

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)}\right]^{-1} = \left[\sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2} Kf_p\right]^{-1}.$$

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).

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Global Robustness → Kf_p's Local Vulnerabilities → C_p's

Local Vulnerability:
Perturbing a specific node k i.e.
$\delta P_{0i} = \delta_{ik}\delta P_0$,
$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2d}[C_1^{-1}(k) - n^{-2}Kf_1]\,,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2md}\frac{(n-1)}{n}\,,$$

$\tau_0 \gg m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \delta P_0^2 \tau_0 [C_2^{-1}(k) - n^{-2}Kf_2]\,,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2}{d}[C_1^{-1}(k) - n^{-2}Kf_1]\,.$$

Global Robustness:
Averaging over an ergodic ensemble of perturbation vectors,
$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\langle \mathcal{P}_1^\infty \rangle = \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2nd} Kf_1\,,$$
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MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

Global Robustness $\rightarrow$ Kf_p's Local Vulnerabilities $\rightarrow$ C_p's

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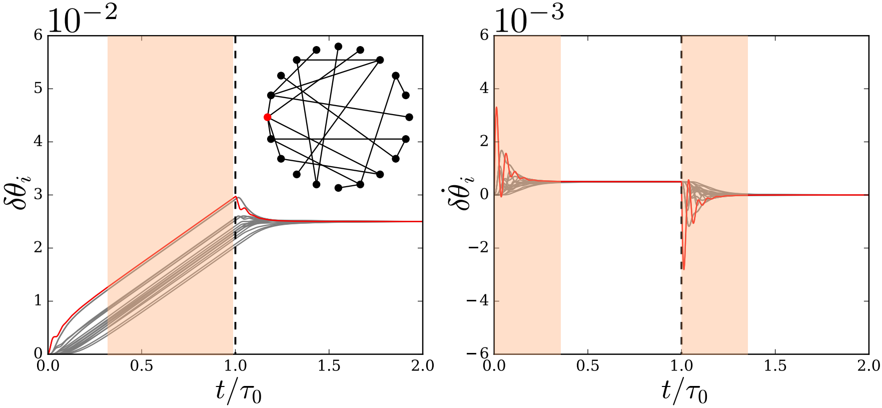

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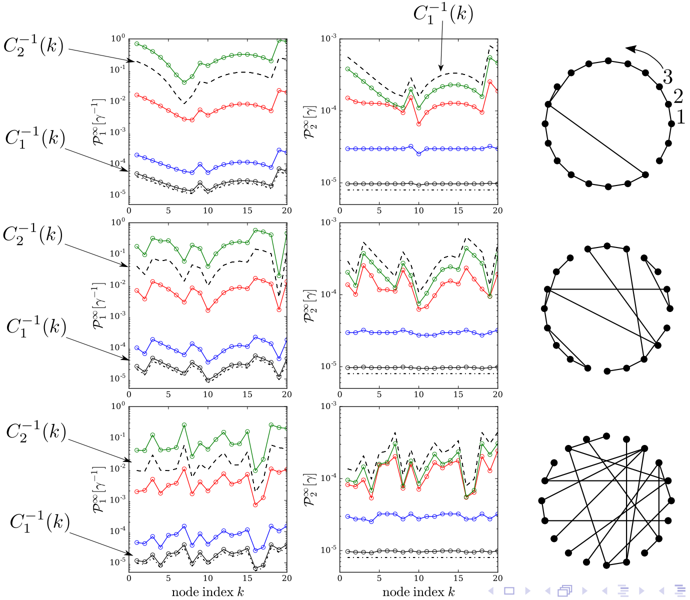

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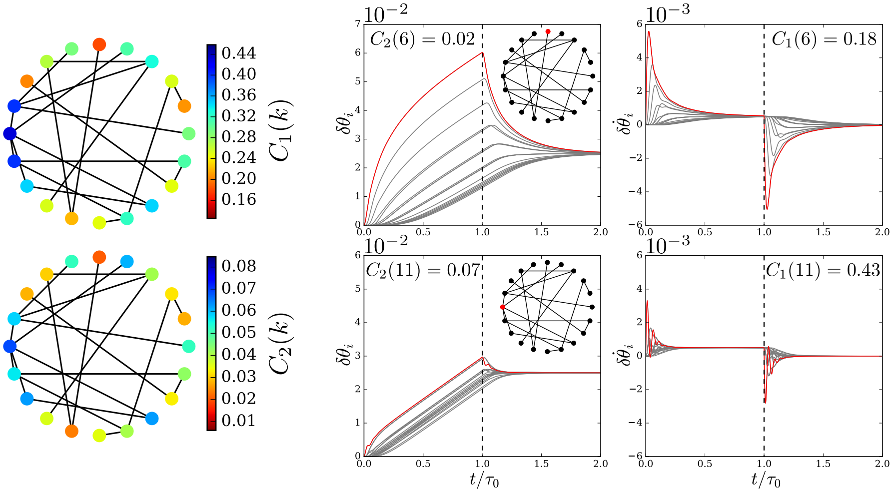

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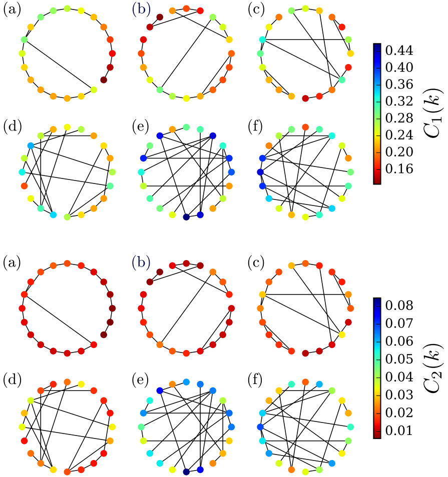

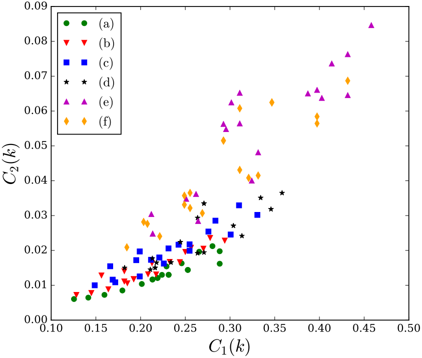

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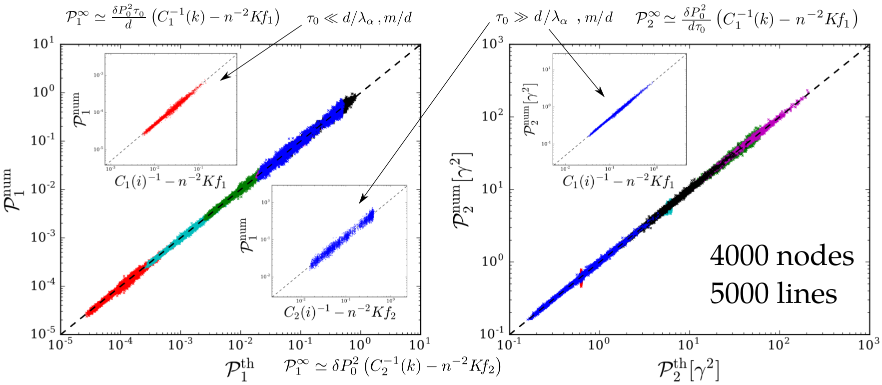

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Physical Realization : European Electrical Grid

$\tau_0 \ll d/\lambda_\alpha \, , m/d$

$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

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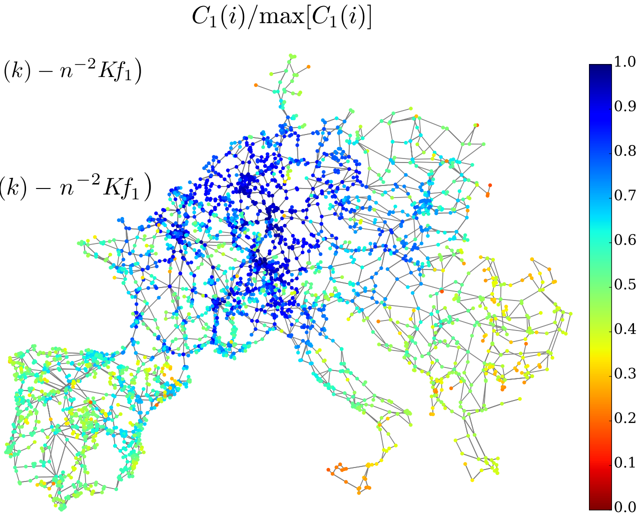

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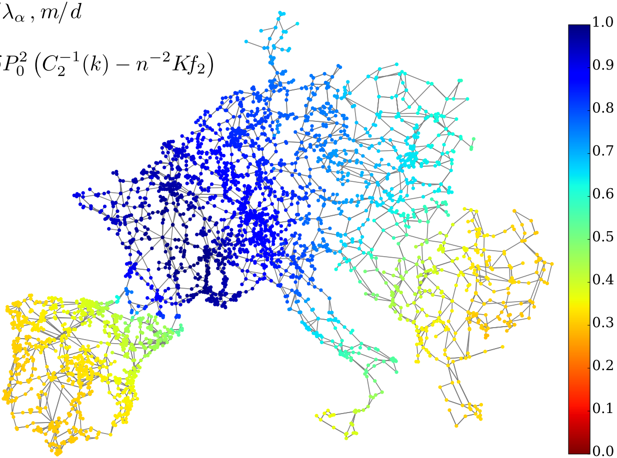

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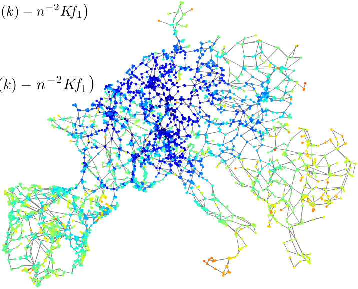

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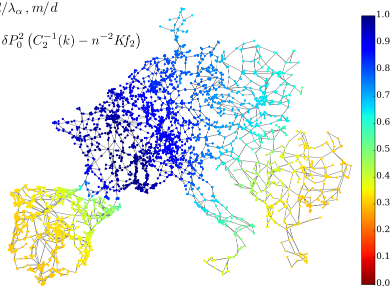

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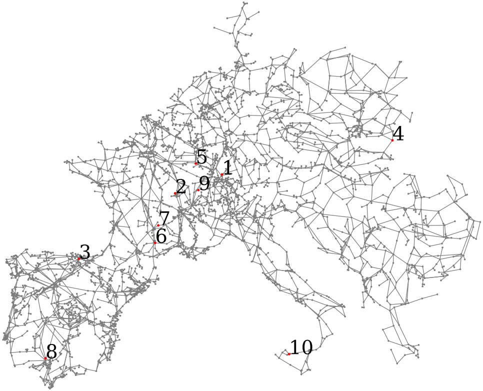

node #	$C_{\rm geo}$	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{\rm num}$	$\mathcal{P}_2^{\rm num}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Ranking of Vulnerabilities

Ranking of the nodes

$\mathrm{LRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\,, & i \neq j\,, \\ \sum_k b_{ik}\,, & i = j\,. \end{cases}$$

$\mathrm{WLRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,, \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,. \end{cases}$$

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

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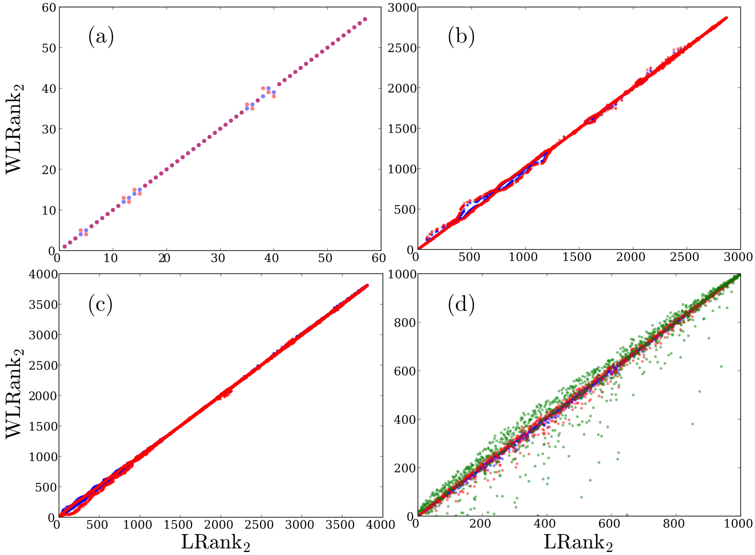

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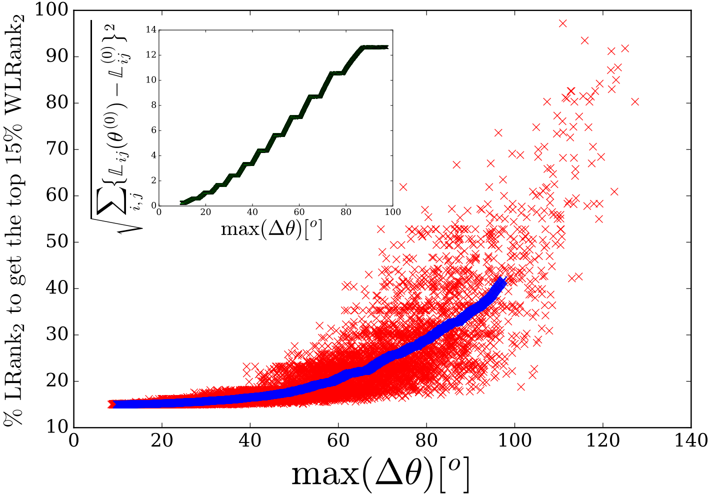

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Summary: Local Vulnerabilities

- Most vulnerable nodes $\rightarrow$ C_1 or C_2 depending on τ_0 and on the dynamical variable of interest.
- Rank the nodes: independent of the operational/synchronous state if $|\Delta\theta| < 30^\circ \rightarrow \mathrm{LRank} \cong \mathrm{WLRank}$.

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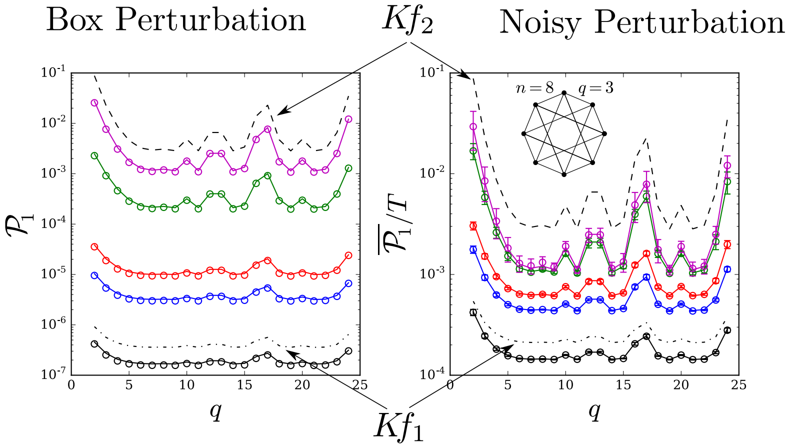
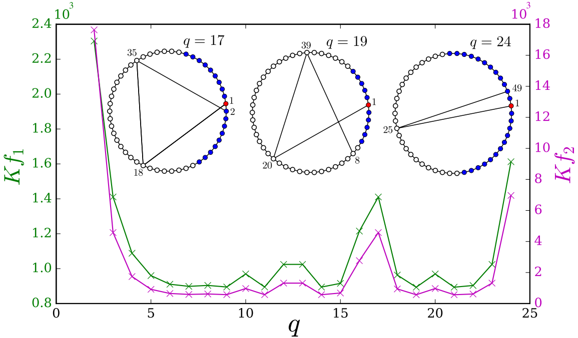
Averaged Global Robustness and Kf_p's: Numerics



MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).

Averaged Global Robustness and Kf_p's: Regular Networks

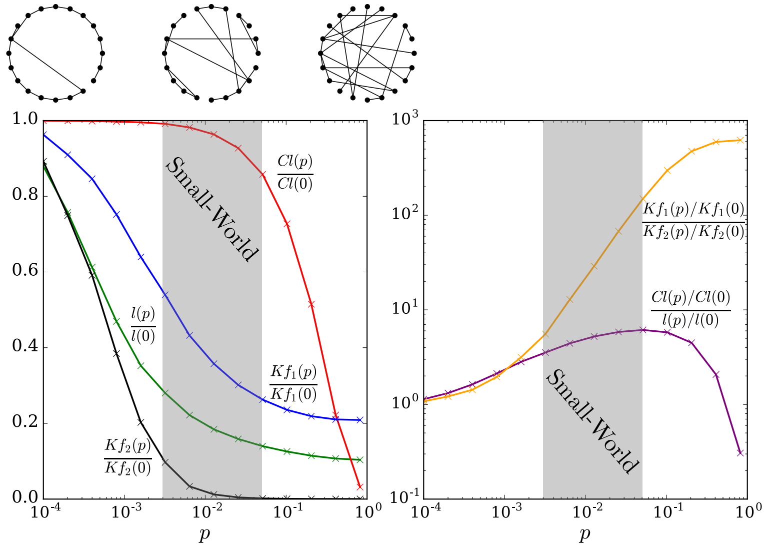


$$Kf_p = \sum_{\alpha \geq 2} [4 - 2\cos(k_\alpha) - 2\cos(qk_\alpha)]^{-p}$$

$$k_\alpha = \frac{2\pi(\alpha - 1)}{n}.$$

MT, Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
MT and Jacquod to appear in *Phys. Rev. E* (2019), arXiv:1905.03582.

Averaged Global Robustness and Kf_p's: Small-World



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Conclusion

Global Robustness

- Generalized Kirchhoff Indices, Kf_p's.

Local Vulnerabilities

- Generalized Resistance Centralities, C_p's.
- Establish a ranking of the nodes.

$\rightarrow$ p depends on which performance measures you are interested in and on the correlation time of the perturbation.

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Inertia

- No effect on performance measures in both asymptotics in τ_0 except for frequencies and short τ_0.

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Line perturbations

$\rightarrow$ *Rate of change of frequency under line contingencies in high voltage electric power networks with uncertainties*, Delabays, MT and Jacquod, arXiv:1906.05698.

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
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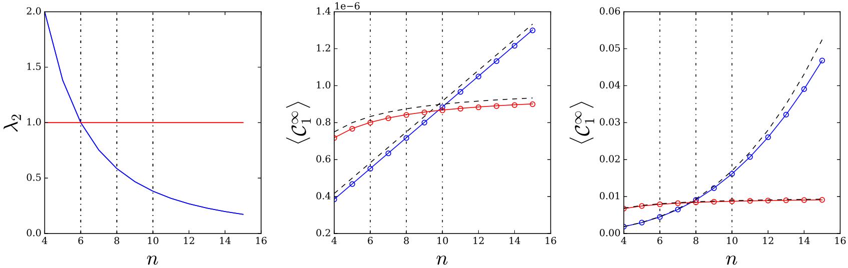
HES-SO Valais-Wallis

Supplemental Material



blue : cycle graph
red : star graph

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

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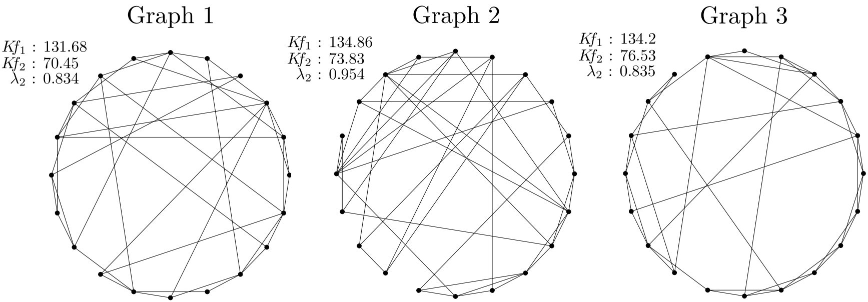

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